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On General Two-Point Continued Fraction Expansions and Padé Tables

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Evelyn Frank's algorithm is applied to the expansion of a pair of non-normal power series into a general M-fraction. The block structure with respect to the M-table for this series-pair is considered taking full account of the blocks formed in the table of Hankel determinants. The M-table is generalized for other types of pairs of power series. In this context, Ramanujan's continued fractions serve as very good examples to illustrate these general continued fraction expansions and the block structure of convergents in the relevant two-point Padé table.

1. INTRODUCTION

Let

$$C(z) = c_0 + c_1 z + c_2 z^2 + \cdots, \qquad c_0 \neq 0 \tag{1.1}$$

and

$$D(z) = d_1 z^{-1} + d_2 z^{-2} + d_3 z^{-3} + \cdots, \qquad d_1 \neq 0 \tag{1.2}$$

be the power series expansions of a function $f(z)$ about the points $z = 0$ and $z = \infty$, respectively. McCabe [17] and McCabe and Murphy [18] have introduced the M-table for the series $C(z)$ and $D(z)$ through M-fractions of the form

$$M(z) = \frac{a_1}{1 + b_1 z +} \frac{a_2 z}{1 + b_2 z +} \frac{a_3 z}{1 + b_3 z +} \cdots \tag{1.3}$$

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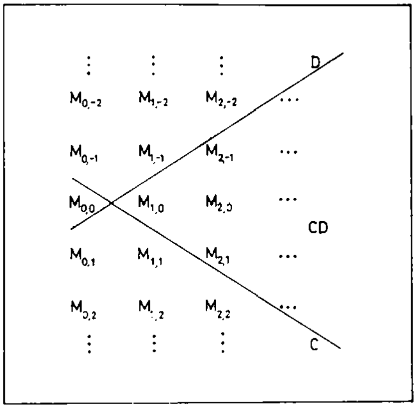

FIG. 1. The M-table.

The above authors formulated certain rhombus rules and also the explicit Hankel determinant expressions for computing the coefficients of M-fractions. These schemes depend heavily on the normality of the power series $C(z)$ and $D(z)$. Cooper *et al.* [6] have generalized the M-table to the non-normal case without imposing restrictions on the coefficients of the series. For full details of basic informative results on the M-table we refer to [17] and [6]. We shall begin our discussion by introducing standard definitions and notations, mainly in accordance with these two important contributions.

We define the M-table as an infinite matrix with (m, n)th entry $M_{m,n}$ (see Fig. 1), $m = 0, 1, 2, ..., n = 0, \pm 1, \pm 2, ...,$ where

$$M_{m,n} = P_{m,n}(z)/Q_{m,n}(z)$$

which is determined by

$$\begin{aligned} C(z)\,Q_{m,n}(z) - P_{m,n}(z) &= O(z^{m+n}), \\ D(z)\,Q_{m,n}(z) - P_{m,n}(z) &= O\ \ (z^{n-1}), \end{aligned} \tag{1.4}$$

where $O(z^k)$ denotes a power series in ascending powers of z beginning with a term in z^k while $O_-(z^k)$ represents a series in descending powers of z. Depending on the correspondence of $M_{m,n}$ with $C(z)$ and $D(z)$, the M-table is divided into three regions as shown in Fig. 1. The entry $M_{0,0}$, identically zero, does not belong to any of the regions. In region CD, $M_{m,n}$ corresponds to the series $C(z)$ and $D(z)$ simultaneously. The constituent polynomials in $M_{m,n}$ are given by

$$Q_{m,n}(z) = \sum_{i=0}^{m} q_i z^i \tag{1.5}$$

and

$$P_{m,n}(z) = \begin{cases} \sum_{j=0}^{n-1} p_j z^j, & \text{in region } C \\ \sum_{j=0}^{m-1} p_j z^j, & \text{in region } CD \\ z^{m+n} \sum_{j=0}^{-n-1} p_j z^j, & \text{in region } D. \end{cases} \tag{1.6}$$

The entries in region C are Padé approximants [1, 2, 11] of the series $C(z)$ with

$$M_{m,n}(z) = [(n-1)/m]_C(z),$$

a rational function whose numerator and denominator polynomials are of degree at most $(n-1)$ and m, respectively. The typical element in region D can be written as

$$M_{m,-n} = E_m^{(n-m)}(z),$$

an element in Wynn's E-array [26] for $D(z)$.

An entry $M_{m,n}(z)$ in the M-table is said to be normal, if it is not equal to any other entry there. An M-table is normal, if every entry in it is normal and the series $C(z)$ and $D(z)$ themselves are said to be normal. The non-normal case studied in [6] and the continued fraction (CF) expansion suggested therein motivated us to consider the problem of constructing CF for a pair of non-normal series.

In Section 2 we consider a class of general M-fractions of the form

$$M(z) = \frac{a_1}{1 + b_1 z^{\beta_1}} - \frac{a_2 z^{\alpha_2}}{1 + b_2 z^{\beta_2}} - \frac{a_3 z^{\alpha_3}}{1 + b_3 z^{\beta_3}} - \cdots \tag{1.7}$$

in which a_n, b_n are complex numbers and α_n, β_n are positive integers such that $\alpha_1 + \alpha_2 + \cdots + \alpha_n < \beta_1 + \beta_2 + \cdots + \beta_n$ (we take $\alpha_1 = 0$ and $\beta_1 = 1$). We also present a method for its construction. The method is based on an algorithm given by Frank [8] for expanding an arbitrary power series into a C-fraction. The block structure of the M-table is considered in Section 3. In Section 4 we consider a general T-table, a generalization of the M-table, for a pair of formal Laurent series (fLs). A T-fraction expansion for a pair of fLs is also given there. Section 5 contains examples illustrating some of the concepts mentioned above. Finally a few concluding remarks are added.

There exist many more two-point Padé tables in the literature. The Laurent–Padé table [3], Chebyshev–Padé table [9, 20], and Colorado table [16] sometimes possess quite interesting structures. In [5] it is shown that the M-table is related to the moment problems. A mixed Padé table and its properties are critically considered in [7]. A two-point Padé table of the brick structure type has been discussed in [15]. Two-point Padé tables for general M-fractions (normal or non-normal) have been studied in [21]. Also a two-point Padé table associated with Schur sequence has been dealt with in [23]. In [24], some useful ideas concerning the structures of such tables for general rational functions are also furnished.

2. Expansion Algorithm for a General M-Fraction

We shall now consider the general M-fraction of the form (1.7). Its nth convergent $P_n(z)/Q_n(z)$ is defined by means of the fundamental recurrence relations

$$\begin{aligned} P_n(z) &= (1+b_n z^{\beta_n}) P_{n-1}(z) - a_n z^{\alpha_n} P_{n-2}(z), \\ Q_n(z) &= (1+b_n z^{\beta_n}) Q_{n-1}(z) - a_n z^{\alpha_n} Q_{n-2}(z), \qquad n \geqslant 2 \end{aligned} \tag{2.1}$$

with

$$P_0 = 0,\ Q_0 = 1,\ P_1 = a_1,\ Q_1 = 1 + b_1 z.$$

Multiplying the first formula of (2.1) by $Q_{n-1}(z)$ and the second by $P_{n-1}(z)$ and then subtracting the latter from the former we obtain

$$P_n Q_{n-1} - Q_n P_{n-1} = a_n z^{\alpha_n}(P_{n-1} Q_{n-2} - Q_{n-1} P_{n-2}). \tag{2.2}$$

Applying this result successively, we end up with

$$P_n Q_{n-1} - Q_n P_{n-1} = a_1 a_2 \cdots a_n z^{\alpha_1 + \alpha_2 + \cdots + \alpha_n}. \tag{2.3}$$

It can be easily shown that the highest power terms occurring in $P_n(z)$ and $Q_n(z)$ are $a_1 b_2 b_3 \cdots b_n z^{\beta_2 + \beta_3 + \cdots + \beta_n}$ and $b_1 b_2 \cdots b_n z^{\beta_1 + \beta_2 + \cdots + \beta_n}$, respectively. The difference of two successive convergents of (1.7) is

$$\begin{aligned} &\frac{P_n(z)}{Q_n(z)} - \frac{P_{n-1}(z)}{Q_{n-1}(z)} \\ &\quad = \frac{P_n(z)\, Q_{n-1}(z) - Q_n(z)\, P_{n-1}(z)}{Q_n(z)\, Q_{n-1}(z)} \\ &\quad = \frac{a_1 a_2 \cdots a_n z^{\alpha_1 + \alpha_2 + \cdots + \alpha_n}}{\{1 + \cdots + (b_1 b_2 \cdots b_{n-1})^2\, b_n z^{2(\beta_1 + \beta_2 + \cdots + \beta_{n-1}) + \beta_n}\}}. \end{aligned} \tag{2.4}$$

Expanding the right side of (2.4) in powers of z and z^{-1}, we have

$$\frac{P_n(z)}{Q_n(z)} - \frac{P_{n-1}(z)}{Q_{n-1}(z)} = a_1 a_2 \cdots a_n z^{\alpha_1+\alpha_2+\cdots+\alpha_n} + \cdots \tag{2.5}$$

and

$$\frac{P_n(z)}{Q_n(z)} - \frac{P_{n-1}(z)}{Q_{n-1}(z)} = \frac{a_1 a_2 \cdots a_n}{(b_1 b_2 \cdots b_{n-1})^2 b_n} \times z^{-\{2(\beta_1+\beta_2+\cdots+\beta_{n-1})+\beta_n-(\alpha_1+\alpha_2+\cdots+\alpha_n)\}} + \cdots, \tag{2.6}$$

respectively.

Equations (2.5) and (2.6) imply that the power series expansions of $P_{n-1}(z)/Q_{n-1}(z)$ about $z=0$ and $z=\infty$ agree with that of $P_n(z)/Q_n(z)$, respectively, up to the terms through $z^{\alpha_1+\alpha_2+\cdots+\alpha_n}$ and $z^{-\{2(\beta_1+\beta_2+\cdots+\beta_{n-1})+\beta_n-(\alpha_1+\alpha_2+\cdots+\alpha_n)\}}$. Hence, by Eqs. (2.1), (2.4), (2.5), and (2.6), the CF (1.7) determines uniquely a pair of power series of the form (1.1) and (1.2). Conversely, given this series-pair there exists a CF of the form (1.7). The uniqueness problem has already been considered in detail [21].

It is important to determine the quantities a_n, α_n, b_n, and β_n of (1.7) from the series-pair. In the case of the nth convergent of (1.7), we have

$$P_n(z) = \sum_{i=0}^{\tau_n} p_i z^i; \qquad \tau_n = \sum_{i=2}^{n} \beta_i; \qquad p_{\tau_n} = a_1 b_2 \cdots b_n \tag{2.7}$$

$$Q_n(z) = \sum_{i=0}^{\sigma_n} q_i z^i; \qquad \sigma_n = \sum_{i=1}^{n} \beta_i; \qquad q_{\sigma_n} = b_1 b_2 \cdots b_n, \qquad q_0 = 1. \tag{2.8}$$

From Eqs. (2.5) and (2.6) it follows that

$$C(z) - P_{n-1}(z)/Q_{n-1}(z) \sim a_1 a_2 \cdots a_n z^{\alpha_1+\alpha_2+\cdots+\alpha_n}, \qquad \text{as} \quad z \to 0$$

$$D(z) - P_{n-1}(z)/Q_{n-1}(z) \sim \frac{a_1 \cdots a_n}{(b_1 \cdots b_{n-1})^2 b_n} z^{-\{2(\beta_1+\cdots+\beta_{n-1})+\beta_n-(\alpha_1+\cdots+\alpha_n)\}}, \qquad \text{as} \quad z \to \infty.$$

By virtue of (2.7) and (2.8) the above equations can be written as

$$C(z)\, Q_{n-1}(z) - P_{n-1}(z) \sim a_1 a_2 \cdots a_n z^{\alpha_1+\cdots+\alpha_n}, \qquad \text{as} \quad z \to 0 \tag{2.9}$$

$$D(z)\,Q_{n-1}(z)-P_{n-1}(z)$$
$$\sim \frac{a_1\cdots a_n}{b_1\cdots b_n}\, z^{-\{(\beta_1+\cdots+\beta_n)-(\alpha_1+\cdots+\alpha_n)\}}, \qquad \text{as} \quad z\to\infty. \tag{2.10}$$

Knowledge of $P_{n-1}(z)$ and $Q_{n-1}(z)$ enables us to determine a_n and α_n from (2.9). Once a_n and α_n are determined, b_n and β_n can be found from (2.10). However, we present here a comparatively more convenient scheme, described in the theorem which follows. In this method, only the coefficients of the concerned series and those of the denominaitor polynomials of convergents are actually involved. The orthogonality structures that exist between the denominators of the convergents of the CF and the coefficients of series have also been revealed. The theorem can as well be used for converting an M-fraction into a pair of power series. For the sake of simplicity, we follow the same matrix format originally adopted by Frank. The Eqs. (2.9) and (2.10) are, indeed, useful for proving the theorem.

THEOREM 1. *Let $C(z)$ and $D(z)$ be a pair of formal power series as given by* (1.1) *and* (1.2), *respectively. Set*

$$a_1=c_0,\ b_1=c_0/d_1,\ D_1=d_1,\ \alpha_1=0,\ \beta_1=s_1=1 \tag{2.11}$$

and let

$$Q_0(z)=1, \qquad Q_1(z)=1+b_1 z. \tag{2.12}$$

For $r=2, 3, \ldots$, construct the polynomials

$$Q_r(z)=1+q_{r,1}z+q_{r,2}z^2+\cdots \tag{2.13}$$

and determine the numbers a_r, b_r and the non-negative integers α_r, β_r by means of the recurrence relations

$$[c_n, c_{n-1}, c_{n-2}, \ldots]\begin{bmatrix} 1 \\ q_{r-1,1} \\ q_{r-1,2} \\ \vdots \end{bmatrix}$$
$$=\begin{cases} 0 & \text{if} \quad \alpha_1+\cdots+\alpha_{r-1}<n<\alpha_1+\cdots+\alpha_r \\ a_1a_2\cdots a_r & \text{if} \quad n=\alpha_1+\alpha_2+\cdots+\alpha_r, \end{cases}$$
$$c_i=-d_{-i} \qquad \text{for} \quad i<0, \tag{2.14}$$

$$[d_{s_r}, d_{s_r+1}, d_{s_r+2}, ...] \begin{bmatrix} 1 \\ q_{r-1,1} \\ q_{r-1,2} \\ \vdots \end{bmatrix} = D_r,$$

$$s_r \geqslant 1 \quad \textit{in such a way that the least value of } s_r \textit{ ensures } D_r \neq 0, \tag{2.15}$$

$$b_r = \frac{a_r}{D_r} \cdot D_{r-1}, \tag{2.16}$$

$$\beta_r = \alpha_r + s_r - s_{r-1}, \tag{2.17}$$

$$Q_r(z) = (1 + b_r z^{\beta_r}) Q_{r-1}(z) - a_r z^{\alpha_r} Q_{r-2}(z). \tag{2.18}$$

Then

$$M(z) = \frac{a_1}{1 + b_1 z^{\beta_1} -} \frac{a_2 z^{\alpha_2}}{1 + b_2 z^{\beta_2} -} \frac{a_3 z^{\alpha_3}}{1 + b_3 z^{\beta_3} -} \cdots. \tag{1.7}$$

The polynomials $Q_r(z)$ *in* (2.18) *are the denominators of the convergents of* (1.7).

Proof. The numerator polynomials of (1.7) are readily defined by (2.1). To prove the first half of the theorem it is sufficient to show that the identity

$$C(z) Q_r(z) - P_r(z) = a_1 a_2 \cdots a_{r+1} z^{\alpha_1 + \alpha_2 + \cdots + \alpha_{r+1}} + \cdots \tag{2.19}$$

is true for all values of r. The path for establishing this relation for any r is exactly the same as that given earlier [8]. With regard to the series $D(z)$ we also follow similar arguments. It is straight forward to verify that

$$D(z) Q_0(z) - P_0(z) = \frac{a_1/b_1}{z^{s_1}} + \cdots \tag{2.20}$$

with $s_r = (\beta_1 + \beta_2 + \cdots \beta_r) - (\alpha_1 + \alpha_2 + \cdots + \alpha_r)$.

Let us assume that

$$D(z) Q_r(z) - P_r(z) = \frac{(a_1 a_2 \cdots a_{r+1})/(b_1 b_2 \cdots b_{r+1})}{z^{s_{r+1}}} + \cdots$$

$$\text{for} \quad r = 0, 1, 2, ..., n. \tag{2.21}$$

The proof of the theorem would be complete, if it were shown that (2.21) holds for $r = n + 1$.

Now

$$
\begin{aligned}
D(z)\,Q_{n+1}(z) - P_{n+1}(z) & \\
&= D(z)[1 + b_{n+1} z^{\beta_{n+1}} Q_n(z) - a_{n+1} z^{\alpha_{n+1}} Q_{n-1}(z)] \\
&\quad - [(1 + b_{n+1} z^{\beta_{n+1}})\, P_n(z) - a_{n+1} z^{\alpha_{n+1}} P_{n-1}(z)] \\
&= (1 + b_{n+1} z^{\beta_{n+1}})[D(z)\,Q_n(z) - P_n(z)] \\
&\quad - a_{n+1} z^{\alpha_{n+1}}[D(z)\,Q_{n-1}(z) - P_{n-1}(z)]
\end{aligned} \tag{2.22}
$$

It is known that $Q_{n+1}(z)$ is of degree $\beta_1 + \beta_2 + \cdots + \beta_{n+1}$ and $P_{n+1}(z)$ is one degree less than that of $Q_{n+1}(z)$. Therefore the conclusion is that all the positive power terms in $D(z)\,Q_{n-1}(z)$ are the same as those in $P_{n-1}(z)$. By (2.15), (2.16), and (2.17), the first negative power term occuring in (2.22) is

$$\frac{(a_1 a_2 \cdots a_{n+2})/(b_1 b_2 \cdots b_{n+2})}{z^{(\beta_1 + \beta_2 + \cdots + \beta_{n+2}) - (\alpha_1 + \alpha_2 + \cdots + \alpha_{n+2})}}, \text{ that is,}$$

$$\frac{(a_1 a_2 \cdots a_{n+2})/(b_1 b_2 \cdots b_{n+2})}{z^{s_{n+2}}}.$$

Consequently (2.21) holds valid for $r = n + 1$, and therefore the theorem is now proved.

3. Block Structure of the M-Table

In [6], it has been proved that equal entries of the M-table form a square block and its order is equal to the excess correspondence of the concerned entry with the two series. With this in view, we shall now consider the formation of blocks in the M-table by convergents of (1.7). The denominator polynomial $Q_n(z)$ of (1.7) is of degree $\beta_1 + \beta_2 + \cdots + \beta_n$. By virtue of our construction, the nth convergent $P_n(z)/Q_n(z)$ represents the entry $M_{\beta_1 + \beta_2 + \cdots \beta_n, 0}$ of the M-table. By (2.5) and (2.6), it matches the first $\alpha_1 + \alpha_2 + \cdots + \alpha_{n+1}$ terms of $C(z)$ and $\{\beta_1 + 2(\beta_2 + \beta_3 + \cdots + \beta_n) + \beta_{n+1} - (\alpha_1 + \alpha_2 + \cdots + \alpha_{n+1})\}$ terms of $D(z)$. But it is normally expected to match only the first $(\beta_1 + \beta_2 + \cdots + \beta_n)$ terms in each of the series $C(z)$ and $D(z)$. Therefore the excess corresponding terms of $P_n(z)/Q_n(z)$ with $C(z)$ and $D(z)$ are

$$\omega_n = \{(\alpha_1 + \alpha_2 + \cdots + \alpha_{n+1}) - (\beta_1 + \beta_2 + \cdots + \beta_n)\} \text{ and}$$

$$\mu_n = \{(\beta_2 + \beta_3 + \cdots + \beta_{n+1}) - (\alpha_1 + \alpha_2 + \cdots + \alpha_{n+1})\}, \text{ respectively.}$$

Hence the total excess correspondence of $P_n(z)/Q_n(z)$ is $\beta_{n+1} - \beta_1$, that is, $\beta_{n+1} - 1$. Thus the order of the block formed by the nth convergent of (1.7) is $\beta_{n+1} - 1$. With respect to the exact position of $M_{\beta_1+\beta_2+\cdots+\beta_n,0}$, the block boundary extends upto ω_n entries below, μ_n entries above, and $(\omega_n + \mu_n)$ entries on the right side.

We shall consider the block structure of the M-table in terms of Hankel determinants of $C(z)$ and $D(z)$. Equation (1.4) gives the system

$$\begin{aligned} c_n q_0 + c_{n-1} q_1 + \cdots + c_{n-m} q_m &= 0 \\ c_{n+1} q_0 + c_n q_1 + \cdots + c_{n-m+1} q_m &= 0 \\ &\;\vdots \\ c_{n+m-1} q_0 + c_{n+m-2} q_1 + \cdots + c_{n-1} q_m &= 0 \\ c_j = -d_{-j} \qquad \text{for} \quad j &< 0 \end{aligned} \tag{3.1}$$

defining denominator of the entry $M_{m,n}$. We have here m equations in $m+1$ unknowns $q_0, q_1, \ldots, q_m$. As in the case of standard Padé approximants [2], we impose the normalization condition $q_0 = 1$ so that the numerator and denominator of $M_{m,n}$ have no common factor. Since all entries of the M-table can be realized as the convergents of M-fractions, this concept of normality is quite convenient. The above system has a unique solution for $q_1, q_2, \ldots, q_m$ if the determinant of the system is non-zero. Associated with the pair of series $C(z)$ and $D(z)$ we have the Hankel determinants $D_{m,n}$ defined by

$$D_{m,n} = \begin{vmatrix} c_{n-m} & c_{n-m+1} & \cdots & c_{n-1} \\ c_{n-m+1} & c_{n-m+2} & \cdots & c_n \\ \vdots & \vdots & & \vdots \\ c_{n-1} & c_n & \cdots & c_{n+m-2} \end{vmatrix}, \qquad c_j = -d_{-j} \text{ for } j < 0$$

$$m = 0, 1, 2, \ldots, \qquad n = 0, \pm 1, \pm 2 \cdots \tag{3.2}$$

The Padé table of a general series, its associated C-table, their block structures, and the various ingredient properties and theorems are all very well known [2, 11, 25]. Since region C of Fig. 1 is a part of both the Padé table and the M-table, the notion of Padé approximant, the Padé table, and the various connected concepts can also be extended to the M-table. Therefore, we can apply the same arguments as we do to the Padé table of a single series. As the properties of the Padé table (region C) and the E-array (region D) are familiar, we shall confine our attention and arguments to the region CD only of the M-table.

To aid our discussion, we shall define the following:

$$\mathbf{q} = \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ \vdots \end{bmatrix}, \qquad \mathbf{p} = \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ \vdots \end{bmatrix},$$

$$R_{m,n} = \begin{bmatrix} c_n & c_{n-1} & \cdots & c_{n-m} \\ c_{n+1} & c_n & \cdots & c_{n-m+1} \\ \vdots & & & \\ c_{n+m-1} & c_{n+m-2} & \cdots & c_{n-1} \end{bmatrix}, \qquad c_j = -d_{-j} \text{ for } j < 0$$

$$S_{m,n} = \begin{bmatrix} c_0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ c_1 & c_0 & 0 & \cdots & 0 & 0 & 0 \\ c_2 & c_1 & c_0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & & & & & \\ 0 & 0 & 0 & \cdots & d_1 & d_2 & d_3 \\ 0 & 0 & 0 & \cdots & 0 & d_1 & d_2 \\ 0 & 0 & 0 & \cdots & 0 & 0 & d_1 \end{bmatrix}.$$

$S_{m,n}$ is an $(m \times m+1)$ order matrix in which each row is made up of either c's or d's only. Of the m rows of $S_{m,n}$ at least n rows are made up of c's when n is positive and d's when n negative. The Eq. (1.4) can then be expressed in the form

$$R_{m,n}\mathbf{q} = \mathbf{0} \tag{3.3}$$

$$S_{m,n}\mathbf{q} = \mathbf{p}. \tag{3.4}$$

Let us consider the table of $D_{m,n}$ determinants, defined by (3.2). The columns and rows are again labelled by m and n, respectively. We name such a table a D-table. The first column elements in this table are just unity and the second column gives the coefficients of the respective series. We now investigate what happens when one of the entries of the D-table vanishes. For definiteness let us assume that $D_{m,n} = 0$ and all other determinants remain non-zero. Consider the following block of four entries of the M-table in the region CD and its corresponding Hankel determinants:

$$\begin{array}{cccc} M_{m-1,n-1} & M_{m,n-1} & \qquad D_{m-1,n-1} & D_{m,n-1} \\ M_{m-1,n} & M_{m,n} & \qquad D_{m-1,n} & D_{m,n} \end{array}$$

(i) When $D_{m,n}=0$, Eq. (3.3) has a solution for $\mathbf{q}$ where $q_0=0$. Then the solution $\mathbf{p}$ given by Eq. (3.4) has $p_0=0$. This shows that the polynomials $P_{m,n}(z)$ and $Q_{m,n}(z)$ of $M_{m,n}$ have the common factor z. After z, is cancelled $M_{m,n}$ reduces just to $M_{m-1,n-1}$.

(ii) When $D_{m,n}=0$, the equation $R_{m,n-1}\mathbf{q}=\mathbf{0}$ has a solution in which $q_m=0$. In the solution given by $S_{m,n-1}\mathbf{q}=\mathbf{p}$ we have $p_{m-1}=0$ so that $M_{m,n-1}=M_{m-1,n-1}$.

(iii) The equations $R_{m-1,n}\mathbf{q}=\mathbf{0}$ and $R_{m-1,n-1}\mathbf{q}=\mathbf{0}$ combined together give a set of m homogeneous equations

$$
\begin{aligned}
c_{n-1}q_0+c_{n-2}q_1+\cdots+c_{n-m}q_{m-1}&=0\\
c_nq_0+c_{n-1}q_1+\cdots+c_{n-m+1}q_{m-1}&=0\\
&\vdots\\
c_{n+m-2}q_0+c_{n+m-3}q_1+\cdots+c_{n-1}q_{m-1}&=0\\
c_j=-d_{-j}\qquad \text{for}\quad j&<0
\end{aligned}
$$

If $D_{m,n}=0$, the above system can possess an infinite number of non-trivial solutions. Hence the solutions offered by the above equations are proportional and so also are the solutions given by $S_{m-1,n-1}\mathbf{q}=\mathbf{p}$ and $S_{m-1,n}\mathbf{q}=\mathbf{p}$. Therefore $M_{m-1,n}=M_{m-1,n-1}$.

Thus the four entries in the block considered above are identically equal to $M_{m-1,n-1}$ simply because of the fact that $D_{m,n}=0$. The condition for $M_{m,n}$ to be distinct from other entries is that the determinants $D_{m,n}$ $D_{m,n+1}$, $D_{m+1,n}$, and $D_{m+1,n+1}$ should be different from zero.

We note that the C- and D-tables are essentially the same. Consequently the block structure of the usual Padé table and that of the M-table are not actually different. However, we present some of its main aspects briefly here. As in the case of the C-table, the D-table can also be calculated recursively by Sylvester's cross-rule formula [11]

$$
\begin{array}{ccc} & .N & \\ .W & .C & .E, \\ & .S & \end{array}\qquad WE=NS-C^2 \tag{3.5}
$$

Now let $D_{m,n+1}$ also be zero in addition to $D_{m,n}$ and further let us take $D_{m+1,n-1}$ and $D_{m-1,n+1}$ as non-zero. If we now take successively the entries $D_{m+1,n}$ and $D_{m+1,n+1}$ of the D-table as the central elements C in (3.5), it follows from the assumptions regarding non-zero elements that $D_{m+1,n}$ and $D_{m+1,n+1}$ are also zero, thus completing a 2×2 block of zeros in the D-table. We have already seen that an isolated zero of the D-table means 2×2 block structure in the M-table. Applying this correspondence to each of the elements of the zero-block in the D-table, we arrive at a 3×3 block structure in the M-table. In general an $r \times r$ zero block in the D-table leads to an $(r+1) \times (r+1)$ array of identical elements in the M-table. This $(r+1) \times (r+1)$ array of equal entries is said to be a block of order r. Frank's theorem [8] on Padé tables can also be extended to the M-table. The explicit version of the theorem is the following:

THEOREM 2. *Let $D_{m,n}$ be as defined in* (3.2). *A set of necessary and sufficient conditions for the M-table for the series $C(z)$ and $D(z)$ to contain the element $M_{m,n}$ as a block of order r can be summarised as*

(i) $D_{m,n} \neq 0$

(ii) $D_{m,n+1} \neq 0$

(iii) $D_{m+1,n} \neq 0$

(iv) $D_{m-r+1,n+r-1} \neq 0$

(v) $D_{m+i,n+j} = 0$ $i, j = 1, 2, ..., r$.

Proof of the above runs similar to that in respect of the said theorem.

4. General T-Fractions and T-Tables

We shall now show how we might, in a quite general case [12–14], apply the method of M-fraction expansions to the construction of T-fractions. The method consists in replacing the given pair of series by another pair. Let

$$C(z) = \sum_{k=0}^{\gamma} c_k^* z^{-k} + c_1 z + c_2 z^2 + \cdots \tag{4.1}$$

and

$$D(z) = \sum_{k=0}^{\mu} d_k^* z^k + d_1 z^{-1} + d_2 z^{-2} + \cdots \tag{4.2}$$

be the given pair of fLs. Let

$$\delta(z) = \delta_1 z + \delta_2 z^2 + \delta_3 z^3 + \cdots \tag{4.3}$$

and

$$\lambda(z) = \lambda_0 + \lambda_1 z^{-1} + \lambda_2 z^{-2} + \cdots \tag{4.4}$$

be the new pair obtained from the original through the relations

$$\delta_i = \begin{cases} c_i - d_i^*, & 1 \leqslant i \leqslant \mu \\ c_i & i > \mu \end{cases} \tag{4.5}$$

$$\lambda_j = \begin{cases} d_0^* - c_0^*, & j = 0 \\ d_j - c_j^*, & 1 \leqslant j \leqslant \gamma \\ d_j, & j > \gamma \end{cases} \tag{4.6}$$

On division by z, this new pair gets standardized to the pair considered in Section 2. Therefore, by applying the theory of M-expansions we can obtain a similar table of rational functions, which we may call $T[\mu, \nu]$-table. These rational functions are the convergents of T-fractions, described below:

$$\begin{aligned} T(z) = {} & \sum_{k=1}^{\mu} d_k^* z^k + \sum_{k=1}^{\gamma} c_k^* z^{-k} + c_0^* + \delta_1 z + \cdots + \delta_j z^j \\ & + \frac{\delta_{j+1} z^{j+1}}{1 + b_1^{(j)} z} \, \frac{a_2^{(j)} z}{+ 1 + b_2^{(j)} z} \, \frac{a_3^{(j)} z}{+ 1 + b_3^{(j)} z +} \cdots, \\ & j \geqslant 0, \text{ with } \delta_0 = c_0^* \end{aligned} \tag{4.7}$$

$$\begin{aligned} T(z) = {} & \sum_{k=1}^{\mu} d_k^* z^k + \sum_{k=0}^{\gamma} c_k^* z^{-k} + \lambda_0 + \lambda_1 z^{-1} + \cdots + \lambda_{j-1} z^{-(j-1)} \\ & - \frac{\lambda_{j-1} z^{-(j-1)}}{1 + b_1^{(-j)} z} \, \frac{a_2^{(-j)} z}{+ 1 + b_2^{(-j)} z +} \cdots, \qquad j \geqslant 1. \end{aligned} \tag{4.8}$$

The coefficients of the above T-fractions are directly calculated by the following recurrence relations used by McCabe [17] for M-fraction expansions.

Initial relations:

$$\begin{aligned} a_1^{(j)} &= 0, & j &= 0, \pm 1, \pm 2, \ldots \\ b_1^{(0)} &= \delta_1 / \lambda_0 & & \\ b_1^{(j)} &= -\delta_{j+1} / \delta_j, & j &> 0 \\ b_1^{(-j)} &= -\lambda_{j-1} / \lambda_j, & j &\geqslant 1. \end{aligned} \tag{4.9}$$

Continuation relations:

$$\left.\begin{aligned} a_i^{(j+1)} \times b_{i+1}^{(j)} &= a_i^{(j)} \times b_i^{(j+1)}, \\ a_i^{(j+1)} + b_i^{(j+1)} &= a_{i+1}^{(j)} + b_i^{(j)}, \\ a_{i+1}^{(-j-1)} \times b_{i+1}^{(-j)} &= a_{i+1}^{(-j)} \times b_i^{(-j-1)}, \\ a_{i+1}^{(-j-1)} + b_i^{(-j-1)} &= a_i^{(-j)} + b_i^{(-j)}. \end{aligned}\right\} \quad i = 1, 2, 3, \ldots, j = 0, 1, 2, \ldots \tag{4.10}$$

Given the pair of fLs $C(z)$ and $D(z)$, Jones and Thron [14] have given the necessary and sufficient conditions for the existence of a general T-fraction

$$T(z)=\sum_{k=1}^{\mu} d_k^* z^k+\sum_{k=0}^{\gamma} c_k^* z^{-k}+\frac{a_1 z}{1+b_1 z+}\,\frac{a_2 z}{1+b_2 z+}\cdots. \tag{4.11}$$

There the explicit expressions for the coefficients of T-fraction have been given in terms of the Hankel determinants. We note, however, that the results could be made much more general.

Let $T_{m,n}$, with $m=0, 1, 2, \ldots$ and $n=0, \pm 1, \pm 2, \ldots$, denote the (m, n)th entry of the $T[\mu, \nu)$-table. Its correspondence with the series $C(z)$ and $D(z)$ is given by

$$C(z)-T_{m,n}=O(z^{m+n+1}) \qquad \text{and} \qquad D(z)-T_{m,n}=O\,(z^{n-m}). \tag{4.12}$$

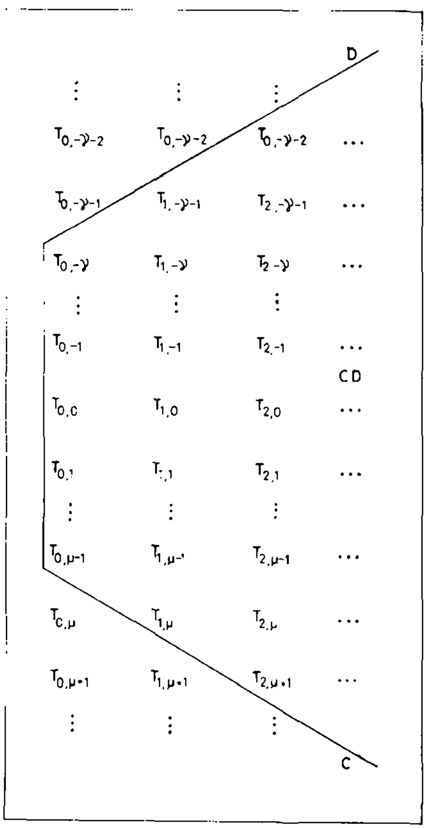

FIG. 2. The $T[\mu, \gamma]$-table.

If we write

$$T_{m,n} = P_{m,n}(z)/Q_{m,n}(z),$$

then $T_{m,n}$ can be determined by

$$\begin{aligned} C(z)\,Q_{m,n}(z) - P_{m,n}(z) &= O(z^{m+n+1}) \\ D(z)\,Q_{m,n}(z) - P_{m,n}(z) &= O\ (z^n). \end{aligned} \tag{4.13}$$

Similarly to the M-table, the $T[\mu, \nu)$-table is divided into three regions as shown in Fig. 2. The entry $T_{0,0}$ equals $\sum_{k=1}^{\mu} d_k^* z^k + \sum_{k=0}^{\nu} c_k^* z^{-k}$. The remaining first column elements represent the partial sums of the correponding δ- or λ-series together with the sum $T_{0,0}$. The denominator coefficients $q_0, q_1, ..., q_m$ of $T_{m,n}$ are determined by a system

$$\begin{array}{lllll} e_{n+1}q_0 + & e_n q_1 & + \cdots + e_{n-m+1} q_m = 0 \\ e_{n+2}q_0 + & e_{n+1} q_1 & + \cdots + e_{n-m+2} q_m = 0 \\ \vdots & \vdots & \qquad\qquad \vdots \\ e_{n+m}q_0 + e_{n+m-1} q_1 + & \cdots + & \quad e_n q_m \quad = 0 \end{array} \tag{4.14}$$

where

$$e_i = \begin{cases} \delta_i, & i \geqslant 1 \\ -\lambda_{-i}, & i \leqslant 0. \end{cases}$$

Therefore, we define the Hankel determinants of the series $C(z)$ and $D(z)$,

$$H_{m,n} = \begin{vmatrix} e_{n-m+1} & e_{n-m+2} & \cdots & e_n \\ e_{n-m+2} & e_{n-m+3} & \cdots & e_{n+1} \\ \vdots & & & \\ e_n & e_{n+1} & \cdots & e_{n+m-1} \end{vmatrix},$$

$$m = 0, 1, 2, ..., \quad n = 0\ \pm 1, \ \pm 2, ... \tag{4.15}$$

with

$$H_{0,n} = 1.$$

The normality of T-table requires the non-vanishing of the Hankel determinants. We shall now present a method for obtaining a general T-fraction expansion from a given pair of fLs.

THEOREM 3. *Let* $C(z)$, $D(z)$, $\delta(z)$ *and* $\lambda(z)$ *be the power series as in*

(4.1)–(4.6). *Set* $\alpha_0 = 0$, $D_0 = 1$, $s_0 = 0$ *and the polynomials* $Q_{-1}(z) = 0$, $Q_0(z) = 1$. *For* $r = 1, 2, 3, \ldots$, *construct the polynomials*

$$Q_r(z) = 1 + q_{r,1} z + q_{r,2} z^2 + \cdots \tag{4.16}$$

and determine the numbers a_r, b_r *and non-negative integers* α_r, β_r *by means of the recurrence relations*

$$[\delta_n, \delta_{n-1}, \delta_{n-2}, \ldots] \begin{bmatrix} 1 \\ q_{r-1,1} \\ q_{r-1,2} \\ \vdots \end{bmatrix}$$

$$= \begin{cases} 0 & \text{if } \alpha_0 + \alpha_1 + \cdots + \alpha_{r-1} < n < \alpha_0 + \alpha_1 + \cdots + \alpha_r \\ a_1 a_2 \cdots a_r & \text{if } n = \alpha_0 + \alpha_1 + \cdots + \alpha_r \end{cases}$$

$$\delta_j = -\lambda_{-j} \text{ for } j \leqslant 0 \tag{4.17}$$

$$[\lambda_{s_r}, \lambda_{s_r+1}, \lambda_{s_r+2}, \ldots] \begin{bmatrix} 1 \\ q_{r-1,1} \\ q_{r-1,2} \\ \vdots \end{bmatrix}$$

$$= D_r,\ s_r \geqslant 0 \text{ in such a way that the least value of } s_r \text{ ensures } D_r \neq 0 \tag{4.18}$$

$$b_r = \frac{a_r}{D_r} \cdot D_{r-1} \quad \text{and} \quad \beta_r = \alpha_r + s_r - s_{r-1} \tag{4.19}$$

$$Q_r(z) = (1 + b_r z^{\beta_r})\, Q_{r-1}(z) - a_r z^{\alpha_r} Q_{r-2}(z). \tag{4.20}$$

Then

$$T(z) = \sum_{k=1}^{\mu} d_k^* z^k + \sum_{k=0}^{\gamma} c_k^* z^{-k} + \frac{a_1 z^{\alpha_1}}{1 + b_1 z^{\beta_1}} - \frac{a_2 z^{\alpha_2}}{1 + b_2 z^{\beta_2}} - \cdots. \tag{4.21}$$

The polynomials $Q_r(z)$ are the denominators of rth convergents of (4.21). The numerator polynomials of the convergents are determined by the recurrence formulae

$$\begin{aligned} P_{-1}(z) &= 1, \\ P_0(z) &= \sum_{k=1}^{\mu} d_k^* z^k + \sum_{k=0}^{\gamma} c_k^* z^{-k} \\ P_r(z) &= P_{r-1}(z)(1 + b_r z^{\beta_r}) - P_{r-2}(z)\, a_r z^{\alpha_r} \qquad r = 1, 2, 3, \ldots \end{aligned} \tag{4.22}$$

The method of proof here is very similar to that of Theorem 1 and so it is omitted.

It is rather important here to take note of one essential fact: whatever the values of μ and γ, the forms of the δ- and λ-series remain unchanged. As a result the forms of Hankel determinants, denominator polynomials of T-tables, nth $(n \geqslant 1)$ partial numerators, and denominators of the relevant CFs remain the same. One may without much difficulty verify that the M and $T[\mu, \gamma)$-tables have the same set of possible block configurations.

5. Examples

The identities of Srinivasa Ramanujan [19],

$$1 + z + \frac{z^2}{1+z^3+} \frac{z^4}{1+z^5+} \cdots$$
$$= \prod_{n=0}^{\infty} \frac{(1-z^{8n+3})(1-z^{8n+5})}{(1-z^{8n+1})(1-z^{8n+7})}, \qquad |z| < 1 \tag{5.1}$$

and

$$1 + \frac{z}{1+z+} \frac{z^2}{1+z^2+} \frac{z^3}{1+z^3+} \cdots$$
$$= \prod_{n=1}^{\infty} (1+z^{2n-1})/(1+z^{2n}), \qquad |z| < 1, \tag{5.2}$$

are the remarkable examples for the general two-point CF expansions considered in this paper. These CFs have been discussed by Gordon [10] and Carlitz [4]. Slater [22] gave combinatorial interpretations to the above identities. All these authors have mainly considered the proof of the identities. We are concerned with much more and shall now consider these CFs from the point of view of Padé approximants. By introducing a factor $(1-z^{8n+8})$ in the numerator and denominator of the infinite product (5.1) and by actual multiplication of the factors, one can get

$$1 + z + \frac{z^2}{1+z^3+} \frac{z^4}{1+z^5+} \cdots$$
$$= \frac{1 - z^3 - z^5 + z^{14} + z^{18} - z^{33} - z^{39} + \cdots}{1 - z - z^7 + z^{10} + z^{22} - z^{27} - z^{45} + \cdots}. \tag{5.3}$$

Denoting the ratio in RHS of (5.3) by $f_0(z)/f_1(z)$, the following recursion relation can be verified:

$$f_{n-1}(z) - (1 + z^{2n-1}) f_n(z) = z^{2n} f_{n+1}(z), \qquad n = 1, 2, 3, \ldots, \tag{5.4}$$

where the leading term in every series is unity. The relations (5.4) lead to the CF expansion (5.3). The CFs (5.1) and (5.2) not only correspond to a single infinite product, as given above, but also give rise to another infinite product expressible as a single series in decreasing powers of z. A nice example for the general M-fraction would be the reciprocal of (5.1),

$$\begin{aligned}\frac{1}{1+z+}\,\frac{z^2}{1+z^3+}\,\frac{z^4}{1+z^5+}\cdots\\ =\prod_{n=0}^{\infty}\frac{(1-z^{8n+1})(1-z^{8n+7})}{(1-z^{8n+3})(1-z^{8n+5})}.\end{aligned}\tag{5.5}$$

From the definition of the series $C(z)$ and $D(z)$ corresponding to the CF(5.5), the coefficients of the series are connected by

$$c_n=d_{n+1},\qquad n=0,1,2,...,\tag{5.6}$$

where

$$\begin{aligned}C(z)=1&-z+z^3-z^4+z^5-2z^7+2z^8-z^9+2z^{11}-3z^{12}\\ &+2z^{13}-2z^{15}+4z^{16}-4z^{17}+4z^{19}-6z^{20}+5z^{21}\\ &-6z^{23}+9z^{24}-6z^{25}+7z^{27}-12z^{28}+9z^{29}-10z^{31}\\ &+16z^{32}-13z^{33}+15z^{35}-22z^{36}+17z^{37}-20z^{39}\\ &+29z^{40}-21z^{41}+25z^{43}-38z^{44}+28z^{45}-\cdots.\end{aligned}\tag{5.7}$$

In view of the symmetry in the coefficients of $C(z)$ and $D(z)$, we may write

$$\begin{aligned}&\frac{1}{1+z+}\,\frac{z^2}{1+z^3+}\,\frac{z^4}{1+z^5+}\cdots\\ &=\left\{\begin{aligned}C(z)&=\prod_{n=0}^{\infty}\frac{(1-z^{8n+1})(1-z^{8n+7})}{(1-z^{8n+3})(1-z^{8n+5})}\\ D(z)&=\frac{1}{z}\prod_{n=0}^{\infty}\frac{\{1-z^{\ (8n+1)}\}\{1-z^{\ (8n+7)}\}}{\{1-z^{\ (8n+3)}\}\{1-z^{-(8n+5)}\}}.\end{aligned}\right.\end{aligned}\tag{5.8}$$

The first few convergents of (5.8) turn out to be

$$\begin{aligned}&\frac{P_1}{Q_1}=\frac{1}{1+z},\qquad \frac{P_2}{Q_2}=\frac{1+z^3}{1+z+z^2+z^3+z^4},\\ &\frac{P_3}{Q_3}=\frac{1+z^3+z^4+z^5+z^8}{1+z+z^2+z^3+2z^4+2z^5+z^6+z^7+z^8+z^9},\ \ldots.\end{aligned}\tag{5.9}$$

$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
$M_{0,-3}$	$M_{1,-3}$	$M_{2,-3}$	$M_{3,-3}$	$M_{4,-3}$	$M_{5,-3}$	$M_{6,-3}$	$M_{7,-3}$	$M_{8,-3}$	$M_{9,-3}$	$\cdots$
$M_{0,-2}$	$M_{1,-2}$	$M_{2,-2}$	$M_{3,-2}$	$M_{4,-2}$	$M_{5,-2}$	$M_{6,-2}$	$M_{7,-2}$	$M_{8,-2}$	$M_{9,-2}$	$\cdots$
$M_{0,-1}$	$M_{1,-1}$	$M_{2,1}$	$M_{3,-1}$	$M_{4,-1}$	$M_{5,-1}$	$M_{6,-1}$	$M_{7,-1}$	$M_{8,-1}$	$M_{9,-1}$	$\cdots$
$M_{0,0}$	$M_{1,0}$	$M_{2,0}$	$M_{3,0}$	$M_{4,0}$	$M_{5,0}$	$M_{6,0}$	$M_{7,0}$	$M_{8,0}$	$M_{9,0}$	$\cdots$
$M_{0,1}$	$M_{1,1}$	$M_{2,1}$	$M_{3,1}$	$M_{4,1}$	$M_{5,1}$	$M_{6,1}$	$M_{7,1}$	$M_{8,1}$	$M_{9,1}$	$\cdots$
$M_{0,2}$	$M_{1,2}$	$M_{2,2}$	$M_{3,2}$	$M_{4,2}$	$M_{5,2}$	$M_{6,2}$	$M_{7,2}$	$M_{8,2}$	$M_{9,2}$	$\cdots$
$M_{0,3}$	$M_{1,3}$	$M_{2,3}$	$M_{3,3}$	$M_{4,3}$	$M_{5,3}$	$M_{6,3}$	$M_{7,3}$	$M_{8,3}$	$M_{9,3}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

FIG. 3. The block structure of convergents of (5.5) in the M-table.

The symmetry in the coefficients of convergents shows that these convergents approximate both the series $C(z)$ and $D(z)$. The block structures formed by the convergents of the M-fraction (5.5) are shown in Fig. 3.

The blocks are due to the vanishing of the set of Hankel determinants of the concerned series:

$$\begin{aligned} D_{1+i,-1+j} &= 0, \qquad i, j = 1, 2 \\ D_{4+i,-2+j} &= 0, \qquad i, j = 1, 2, 3, 4 \\ D_{9+i,-3+j} &= 0, \qquad i, j = 1, 2, ..., 6 \\ &\vdots \end{aligned}$$

The vanishing of the above determinants can be verified with respect to the series (5.6) and (5.7).

The CF (5.1) emerges as an excellent example of the general T-fraction

$$1 + z + \frac{z^2}{1+z^3+} \frac{z^4}{1+z^5+} \cdots = \begin{cases} C(z) \\ D(z), \end{cases} \tag{5.10}$$

where $C(z)$ and $D(z)$ will have to be identified with the series (4.1) and (4.2), respectively. The coefficients of these series are related through

$$c_0^* = d_1^*, \qquad c_1 = d_0^*$$

and (5.11)

$$c_{n+1} = d_n, \qquad n = 1, 2, 3, ...,$$

where

$$\begin{aligned} C(z) = 1 &+ z + z^2 - z^5 - z^6 + z^8 + 2z^9 + z^{10} - 2z^{12} - 3z^{13} - 2z^{14} \\ &+ 3z^{16} + 4z^{17} + 4z^{18} - 4z^{20} - 6z^{21} - 5z^{22} + 5z^{24} \\ &+ 9z^{25} + 6z^{26} - 8z^{28} - 12z^{29} - 9z^{30} + 12z^{32} + 16z^{33} \\ &+ 13z^{24} - 14z^{36} - 22z^{37} - 17z^{38} + 18z^{40} + 29z^{41} \\ &+ 21z^{42} - 26z^{44} - 38z^{45} - \cdots. \end{aligned} \tag{5.12}$$

The block configurations of convergents of (5.10) which are actually the reciprocals of (5.9) in the T-table would be as shown in Fig. 4.

The following are zero-blocks of Hankel determinants of the series $C(z)$ and $D(z)$ associated with the above block structure:

$$\begin{aligned} H_{i,-1+j} &= 0, \qquad i, j = 1, 2 \\ H_{3+i,-2+j} &= 0, \qquad i, j = 1, 2, 3, 4 \\ H_{8+i,-3+j} &= 0, \qquad i, j = 1, 2, ..., 6 \\ &\vdots \end{aligned}$$

The above assertion can be checked with respect to the series (5.11) and (5.12).

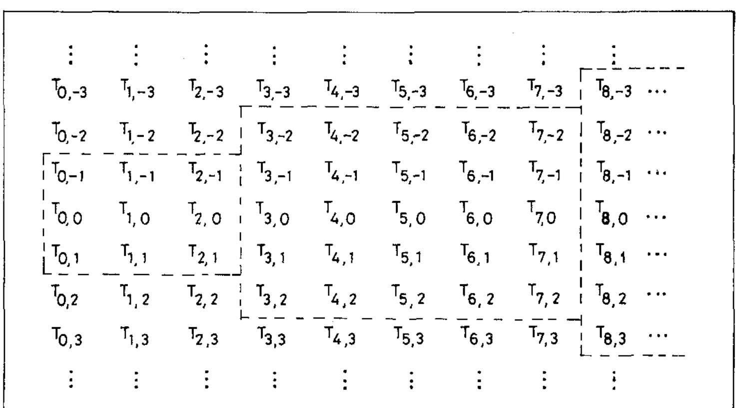

FIG. 4. The block structure of convergents of (5.10) in the $T[1, 0]$-table.

The CF (in 5.2),

$$1+\frac{z}{1+z}+\frac{z^2}{1+z^2}+\frac{z^3}{1+z^3}+\cdots$$
$$=\begin{cases} C(z)=\prod_{n=1}^{\infty}\dfrac{(1+z^{2n+1})}{(1+z^{2n})} \\ D(z)=2\prod_{n=1}^{\infty}\dfrac{(1+z^{-2n})}{\{1+z^{-(2n-1)}\}}, \end{cases} \tag{5.13}$$

serves as an example of T-fraction ($\mu=0$, $\gamma=0$) given in Section 4. The power series $C(z)$ and $D(z)$ associated with CF (5.13) are

$$\begin{aligned} C(z)=1&+z-z^2+z^4-z^6-z^7+2z^8+z^9-2z^{10}+2z^{12} \\ &+z^{13}-3z^{14}-4z^{15}+4z^{16}+2z^{17}-5z^{18}-2z^{19}+5z^{20} \\ &+2z^{21}-6z^{22}-3z^{23}+8z^{24}+4z^{25}-9z^{26}-\cdots \end{aligned}$$

$$\begin{aligned} D(z)=2(1&-z^{-1}+2z^{-2}-3z^{-3}+4z^{-4}-6z^{-5}+9z^{-6}-12z^{-7}+16z^{-8} \\ &-22z^{-9}+29z^{-10}-38z^{-11}+50z^{-12}-64z^{-13}+82z^{-14} \\ &-105z^{-15}+132z^{-16}-166z^{-17}+208z^{-18}-258z^{-19} \\ &+320z^{-20}-395z^{-21}+484z^{-22}-592z^{-23}+722z^{-24} \\ &-876z^{-25}+\cdots). \end{aligned}$$

The blocks of convergents of (5.13), namely,

$$\frac{P_1}{Q_1}=\frac{1+2z}{1+z},\qquad \frac{P_2}{Q_2}=\frac{1+2z+2z^2+2z^3}{1+z+2z^2+z^3},$$
$$\frac{P_3}{Q_3}=\frac{1+2z+2z^2+4z^3+4z^4+2z^5+2z^6}{1+z+2z^2+3z^3+2z^4+2z^5+z^6},\ldots.$$

are shown in Fig. 5.

Since the set of Hankel determinants

$$\begin{array}{ll} H_{2,1} & \\ H_{3+i,j}, & i,j=1,2 \\ H_{6+i,j}, & i,j=1,2,3 \\ \vdots & \end{array}$$

⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	
$T_{0,-1}$	$T_{1,-1}$	$T_{2,-1}$	$T_{3,-1}$	$T_{4,-1}$	$T_{5,-1}$	$T_{6,-1}$	$T_{7,-1}$	⋯
$T_{0,0}$	$T_{1,0}$	$T_{2,0}$	$T_{3,0}$	$T_{4,0}$	$T_{5,0}$	$T_{6,0}$	$T_{7,0}$	⋯
$T_{0,1}$	$T_{1,1}$	$T_{2,1}$	$T_{3,1}$	$T_{4,1}$	$T_{5,1}$	$T_{6,1}$	$T_{7,1}$	⋯
$T_{0,2}$	$T_{1,2}$	$T_{2,2}$	$T_{3,2}$	$T_{4,2}$	$T_{5,2}$	$T_{6,2}$	$T_{7,2}$	⋯
$T_{0,3}$	$T_{1,3}$	$T_{2,3}$	$T_{3,3}$	$T_{4,3}$	$T_{5,3}$	$T_{6,3}$	$T_{7,3}$	⋯
$T_{0,4}$	$T_{1,4}$	$T_{2,4}$	$T_{3,4}$	$T_{4,4}$	$T_{5,4}$	$T_{6,4}$	$T_{7,4}$	⋯
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	

FIG. 5. The blocks of convergents of (5.13) in the $T[0, 0]$-table.

happen to be zero, we obviously have certain block structures in the $T[0, 0]$-table also. The convergents of Ramanujan's CFs are the approximants of two power series simultaneously, but they are not two-point Padé approximants. If the concerned two series represent the same function, then these CFs will possibly yield two point Padé approximants to the function. However, they do serve as good examples illustrating many of the aspects considered in the present paper.

6. CONCLUDING REMARKS

A working procedure has been provided here for the construction of a general two-point continued fraction from a given pair of non-normal power series. This work was really motivated by our deep desire to study non-normal approximants using continued fractions as essential tools. We do feel that Evelyn Frank's algorithm for non-normal power series with integer and rational coefficients could be the best method available at least at present for obtaining two-point continued fraction expansions. We further observe that all the zeros and poles of the convergents of the general M-fraction given in Section 5 lie very near the circumference of the circle $|z| = 1$. Further study on the significance of this phenomenon and other aspects associated with the block sizes in the Padé table is indeed desirable. In any case the results achieved so far concerning the simultaneous approximation of two series by rational functions and their immediate connections to the Padé approximants have been found to be of considerable value as well as interest.

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